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COUNTER-BALANCED SUSPENSION SYSTEMS: RIGID AND FLEXIBLE (EXTENSIONS OF THE ATWOOD'S MACHINE)

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The simplest counter-balanced suspension system is the classic Atwood's machine. The most ambitious counter-balanced system envisaged inthis report consists of a lattice-type flexible structure suspended over pulleys, no longer considered frictionless, by a set of wires, no longer inextensible, and counter-balanced by an identical structure. Since the two structures do counter-balance each other, the net gravitational effect, at least at the points at which the wires attach to the bodies, is null. As might be imagined, there is a host of possibilities between these extremes. In this report, we record some of these and indicate how the motion of each may be analyzed. Not all systems described are studied in depth; for those that have been, greater detail is given.

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List of Symbols

m_1 = mass 1

m_2 = mass 2

m_1 = reduced mass 1 = $\frac{m_1}{m_1 + m_2}$

m_2 = reduced mass 2 = $\frac{m_2}{m_1 + m_2}$

t = real time

t = pseudo–time

θ_0 = initial angular displacement

$\dot{\theta}_0$ = initial angular velocity

ρ = length of pendulum arm

θ = θ = angular displacement of pendulum arm; two symbols are used to make the notation uniform

$r = \rho/g$ = length of pendulum arm in a system of units in which g = 1

ω = a parameter relating t and t; eventually shown to be a factor specifying frequency

ϵ = a parameter used to establish asymptotic expansion

r_i = a term in the expansion of r

θ_i = a term in the expansion of θ

ω_i = a term in the expansion of ω

μ_i = a term in the expansion of $\mathbf{m}_1$

r_0 = initial length of pendulum arm in system of units for which g = 1

k = a parameter relating to frequency in the system in which g = 1; $k^2 = \omega_0/r_0$

Contents

Section I

Introduction

The Structural Vibration and Acoustics Branch of the US Air Force Flight Dynamics Laboratory (WRDC/FIBG) has initiated a program to study the dynamics and control of Large Space Structures (LSS). This Large Space Structures Technology Program (LSSTP) is intended to enable the Flight Dynamics Laboratory to instrument, test, and analyze large space structures on the ground in order to predict their behavior in space. Since the testing is to be done in a ground based laboratory, i.e. under 1–g acceleration, the experiments must be designed so as to counteract this gravitational effect. One proposal is to use soft suspension systems. Long cables can provide pendulum support with low frequency for horizontal motion, but, for a pendulum, the restraint in the vertical direction is rigid. One way to provide soft restraint vertically, while maintaining the pendulum approach, is to counter–balance the test model.

The simplest counter–balanced suspension system is the classic Atwood's machine.[1] This consists of two masses m_1 and m_2 connected by an inextensible, flexible wire of negligible mass, draped over a frictionless, massless pulley, which in turn is rigidly suspended from an overhead support. The motion of the two masses is assumed to be constrained to the vertical direction: either there is no motion, the situation that occurs when the two masses are equal and there is no initial velocity, or, as one mass rises the other falls. The most ambitious counter–balanced system envisaged in this report consists of a lattice–type flexible structure suspended over pulleys, no longer considered frictionless, by a set of wires, no longer inextensible, and counter–balanced by an identical structure. Since the two structures do counter–balance each other, the net gravitational effect, at least at the points at which the wires attach to the bodies, is null.

As might be imagined, there is a host of possibilities between these extremes. In this report we record some of these and indicate how the motion of each may be analyzed. We have not studied in depth all systems to be described; for those that we have, greater detail will be given.

The theories and methods used come from Lagrangian mechanics, linear elastic theory, the calculus of variations, non–linear differential equations, numerical methods for the solution of algebraic–differential equations, and perturbation theory. Where references are appropriate, these will be offered; the report otherwise reflects the thoughts of the author.

Section II

Statement of the Problem

II.1.a The classical Atwood's machine is depicted in Figure 1.a. The analysis of this device is included only because it serves as a guide to the logic and methods used in subsequent problems.

Using the symbols of Figure 1.a, the kinetic energy is:

$$T = (\tfrac{1}{2})m_1\dot{z}_1^{\,2} + (\tfrac{1}{2})m_2\dot{z}_2^{\,2} \tag{1}$$

and the potential energy is:

$$V = m_1gz_1 + m_2gz_2 \tag{2}$$

The Lagrangian function is:

$$L = T - V \tag{3}$$

There is, however, a constraint, namely, that the length of the wire joining the two masses is constant:

$$\Phi \equiv (h-z_1) + \pi R + (h-z_2) = \ell^* = \text{length of wire} \tag{4}$$

The modified Lagrangian function is:

$$\mathcal{L} = T - V + \lambda(t)\Phi \tag{5}$$

and where $\lambda(t)$ is a function of the time, t.

Hamilton's principle implies:

$$\delta \int_{t_0}^{t_1} \mathcal{L}\, dt = 0 \tag{6}$$

which in turn yields the Euler–Lagrange equations:

$$m_1\ddot{z}_1 + m_1g - \lambda = 0 \tag{7}$$

$$m_2\ddot{z}_2 + m_2g - \lambda = 0 \tag{8}$$

to which we must adjoin the constraint condition: $\Phi = \ell^*$.

With this done, we eliminate λ between Equations (7) and (8) and obtain, using Equation (4):

$$\ddot{z}_1 = \left[\frac{m_2 - m_1}{m_2 + m_1}\right] g \qquad (9)$$

$$\ddot{z}_2 = \left[\frac{m_1 - m_2}{m_1 + m_2}\right] g \qquad (10)$$

When these equations are integrated, we obtain:

$$z_1(t) = \tfrac{1}{2}\left[\frac{m_2 - m_1}{m_2 + m_1}\right] gt^2 + \dot{z}_1(0)t + z_1(0) \qquad (11)$$

$$z_1(t) = \tfrac{1}{2}\left[\frac{m_1 - m_2}{m_1 + m_2}\right] gt^2 + \dot{z}_2(0)t + z_2(0) \qquad (12)$$

The values for $z_1(0)$ and $z_2(0)$ are quite arbitrary and depend on the configuration, i.e. the height h, the radius R and the length of the connecting wire. The values for $\dot{z}_1(0)$ and $\dot{z}_2(0)$ are not arbitrary since $\dot{z}_1(0) + \dot{z}_2(0) = 0$, i.e. as one mass rises the other must fall at the same speed, both initially and for all future time.

II.1.b The modified Atwood's machine described in this section is depicted in Figure 1.b. The method of solution parallels that given above for the classical case.

The kinetic energy is:

$$T = (\tfrac{1}{2})m_1\dot{z}_1^{\,2} + (\tfrac{1}{2})m_2\dot{z}_2^{\,2} \qquad (1)$$

and the potential energy is:

$$V = m_1 g z_1 + m_2 g z_2 \qquad (2)$$

We could proceed and introduce a constraint equation; it would be:

$$\frac{z_1(t) - z_1(0)}{R_1} + \frac{z_2(t) - z_2(0)}{R_2} = 0 \qquad (3)$$

For variety, we circumvent the use of the constraint by reducing the expressions for the kinetic and potential energies (written above as functions of z_1 and z_2 and their time derivatives) to functions of the single variable θ and its time derivative. The transformation that accomplishes this is:

$$z_1 = z_1(0) - R_1\theta \tag{4}$$

$$z_2 = z_2(0) - R_2\theta \tag{5}$$

so that the kinetic energy becomes:

$$T = (\tfrac{1}{2})(m_1R_1{}^2 + m_2R_2{}^2)\,\dot{\theta}^2 \tag{6}$$

and the potential energy becomes:

$$V = m_1gz_1(0) + m_2gz_2(0) + (m_2R_2 - m_1R_1)g\theta \tag{7}$$

The Lagrangian function is:

$$L = T - V \tag{8}$$

and using Hamilton's principle:

$$\delta\int_{t_0}^{t_1} L\,dt = 0 \tag{9}$$

We are led to the Euler–Lagrange equation:

$$\ddot{\theta} = \left[\frac{m_1R_1 - m_2R_2}{m_1R_1{}^2 + m_2R_2{}^2}\right]g \tag{10}$$

In terms of z_1 and z_2, we have:

$$\ddot{z}_1 = \frac{m_2R_2 - m_1R_1}{m_2R_2^2 + m_1R_1{}^2}R_1g \tag{11}$$

$$\ddot{z}_2 = \frac{m_1R_1 - m_2R_2}{m_1R_1^2 + m_2R_2{}^2}R_2g \tag{12}$$

It is obvious that if $R_1 = R_2$, Equations (11) and (12) reduce to Equations (9) and (10) of the previous section [II.1.a].

We assume that the interested reader can integrate and interpret both the first and second integrals of these equations.

II.1.c An idealization of an Atwood's machine wherein the wire connecting the two masses is elastic is given in Figure 1.c. The following additional notation is used:

A = cross-sectional area of the wire
E = Young's modulus

ℓ = length of the unstressed wire
k = elastic constant of the wire = $\frac{AE}{\ell}$

Again, using the nomenclature of that figure, the kinetic energy is:

$$T = (\tfrac{1}{2})m_1\dot{z}_1^{\,2} + (\tfrac{1}{2})m_2\dot{z}_2^{\,2} \tag{1}$$

and the potential energy is:

$$V = m_1gz_1 + m_2gz_2 + V_s \tag{2}$$

and where V_s is the strain energy in the wire. V_s is found, assuming Hooke's law, as follows:

If ℓ is the length of the unstretched wire, i.e. the length before the masses m_1 and m_2 are attached, then the strain after the system is in motion is:

$$s = (2h + \pi R - \ell) - (z_1+z_2) \tag{3}$$

The strain energy is:

$$V_s = (\tfrac{1}{2})ks^2 \tag{4}$$

and k is the spring constant for the wire. Again, the Lagrangian is:

$$L = T - V \tag{5}$$

and Hamilton's principle dictates:

$$\delta \int_{t_1}^{t_1} L\, dt = 0 \tag{6}$$

The Euler–Lagrange equations lead to:

$$m_1\ddot{z}_1 + k(z_1 + z_2) = k[z_1(0) + z_2(0)] - m_1g \tag{7}$$

$$m_2\ddot{z}_2 + k(z_1 + z_2) = k[z_1(0) + z_2(0)] - m_2g \tag{8}$$

which when integrated become:

$$z_1 = m_2(A \cos \sigma t + B \sin \sigma t) + \frac{1}{2}\left[\frac{m_2 - m_1}{m_2 + m_1}\right]gt^2 + v_0 t + z_1^* \tag{9}$$

$$z_2 = m_1(A \cos \sigma t + B \sin \sigma t) + \frac{1}{2}\left[\frac{m_1 - m_2}{m_1 + m_2}\right]gt^2 - v_0 t + z_2^* \tag{10}$$

where A and B are constants of integration, v_0 is an initial velocity,

$$\sigma^2 = k\,\frac{m_1 + m_2}{m_1 m_2} \tag{11}$$

and

$$z_1^* + z_2^* = z_1(0) + z_2(0) - \left[\frac{2 m_1 m_2}{m_1 + m_2}\right]\left[\frac{l_1^{\bullet} + l_2^{\bullet}}{AE}\right] g \tag{12}$$

and where:

$l_i^{\bullet}$ = length of the unstretched wire as measured from the top of the pulley (point P) to the point where the mass m_i is attached (i = 1,2).

We infer from the above equations that the motion of the two masses m_1 and m_2 in an Atwood's machine in which the two masses are connected by an elastic wire consists of two "normal" modes. The first is the motion that would occur if the wire were inelastic. The second may be interpreted as each mass executing simple harmonic motion, the two harmonic motions being in phase, with the amplitude of the motion of mass m_1 being proportional to the mass m_2 and the amplitude of the motion of the mass m_2 being proportional to m_1.[2]

II.1.d With Friction on the Bearing. In this section, the equations of motion for an Atwood's machine for which there is friction on the bearing are derived. Referring to Figure 1.d, T_1 is the tension in the wire that joins the mass m_1 to the pulley and T_2 is the tension in the wire that joins m_2 to the pulley. Also from the figure, it is clear that:

$$z_1 + z_2 = 2h + \pi R - l = \text{constant} \tag{1}$$

where l = length of wire joining the two masses. Differentiating Equation (1) with respect to time yields:

$$\dot{z}_1 + \dot{z}_2 = 0 \tag{2}$$

and

$$\ddot{z}_1 + \ddot{z}_2 = 0 \tag{3}$$

The force acting on m_1 is:

$$T_1 = m_1 g + m_1 \ddot{z}_1 \tag{4}$$

and on m_2:

$$T_2 = m_2 g + m_2 \ddot{z}_2 \tag{5}$$

If $m_2 > m_1$, so that the direction of motion is as depicted in the figure, i.e. clockwise, the forces balance out as:

$$T_2 = T_1 + F_B \tag{6}$$

where F_B is the force due to friction on the bearing. Denoting the coefficient of friction by μ, the total frictional force is assumed to be:

$$F_B = \mu(T_1 + T_2) \tag{7}$$

Thus, Equation (6) becomes, using Equations (3), (4), and (5),

$$T_2 = T_1 + \mu(T_1 + T_2) \tag{8}$$

or

$$\ddot{z}_2 = \frac{(m_1 - m_2) + \mu(m_1 + m_2)}{(m_1 + m_2) + \mu(m_1 - m_2)} g \tag{9}$$

If the coefficient of friction of a particular bearing is unknown, μ may be found by solving for it in Equation (9). A simple experiment, using known masses m_1 and m_2, a known value of g and determining $\ddot{z}_2$, enables one to calculate μ.

II.2.a An Atwood's pendulum is defined as an Atwood's machine in which one of two masses is allowed to swing as a pendulum while the other remains constrained to move only in the vertical direction. The pendulum motion of the one mass induces a varying tension in the connecting wire; this, in turn, produces motion in the second mass. It is shown that this motion can be made periodic if the ratio of the two masses and the dependency of this ratio on the initial conditions are chosen as to be prescribed in this report. If this condition is not met, the motion consists of the superposition of two motions. The first is motion in a constant gravitational field where the effective "gravity" is kg; the factor k is determined explicitly. The second is the periodic motion that is the central theme of this section of the report. During the course of the analysis, the fundamental frequency of the periodic motion is determined. It is shown to be slightly higher than the frequency of a pendulum of comparable length swinging in the earth's gravitational field; the factor is given explicitly. This work is restricted to the extent that *small angle approximations* are introduced initially for trigonometric functions.

The geometry of the configuration studied is given in Figure 2. From this geometry, assuming that the masses of the pulley and the wire are negligible and that the radius of the pulley may be neglected also, the differential equations describing the motion of the mass m_2 are:

$$\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} + g \sin \theta = 0 \tag{1}$$

$$(m_1 + m_2)\ddot{\rho} - m_2\rho\dot{\theta}^2 + m_1 g - m_2 g \cos \theta = 0 \tag{2}$$

where dots indicate differentiation with respect to the time t. Equations (1) and (2) may be simplified slightly if we observe that by letting $\rho = \ell g$, both equations contain g as a factor and we may divide through by it. This is equivalent to choosing units so that $g = 1$. Furthermore, by using the small angle approximation, these equations then become:

$$\ell\ddot{\theta} + 2\dot{\ell}\dot{\theta} + \theta = 0 \tag{3}$$

$$(m_1 + m_2)\ddot{\ell} - m_2\ell\dot{\theta}^2 + (m_1 - m_2) + \tfrac{1}{2} m_2\theta^2 = 0 \tag{4}$$

Case i) Constant ℓ

We prove now that there is only one solution of Equations (3) and (4) for which $\ell = \ell_0 =$ constant. Actually, we shall show that assuming that $\ell =$ constant leads to a contradiction. Using this assumption, Equations (3) and (4) become

$$\ell_0\ddot{\theta} + \theta = 0 \tag{3.1}$$

$$m_2\ell_0\dot{\theta}^2 - (m_1 - m_2) - \tfrac{1}{2} m_2\theta^2 = 0 \tag{4.1}$$

Differentiate Equation (4.1) with respect to the time t and divide through by m_2 to get:

$$2\ell_0\ddot{\theta}\dot{\theta} - \theta\dot{\theta} = 0 \tag{5}$$

If $\dot{\theta} \equiv 0$, then Equation (3.1) implies that $\theta \equiv 0$ and Equation (4.1) then implies that $m_1 = m_2$, the classic Atwood's machine case with equal masses. For this classic example there is indeed a solution $\ell = \ell_0$, a constant. If $\dot{\theta} \not\equiv 0$, then Equation (5) implies that

$$2\ell_0\ddot{\theta} = \theta \tag{6}$$

Equations (3.1) and (6) are incompatible. There is, therefore, no solution of Equations (3) and (4), other than the classic Atwood's machine, for which ℓ is constant.

We proceed now to discover the relationship between m_1 and m_2 for which $\ell(t)$ and $\theta(t)$ are periodic solutions of Equations (3) and (4).

Case ii) The General Case.

The perturbation technique requires that Equations (3) and (4) be modified and rewritten as follows:

$$\ddot{\boldsymbol{r}} = \epsilon[\mathbf{m}_2 \boldsymbol{r} \dot{\theta}^2 - (\mathbf{m}_1 - \mathbf{m}_2) - \tfrac{1}{2}\, \mathbf{m}_2 \theta^2] \tag{7}$$

$$\boldsymbol{r}\ddot{\theta} + \theta = -2\dot{\boldsymbol{r}}\dot{\theta} \tag{8}$$

where we have introduced a parameter ϵ and written $\mathbf{m}_1 = m_1/(m_1 + m_2)$ and $\mathbf{m}_2 = m_2/(m_1 + m_2)$. In Equations (7) and (8), we change the time variable $\boldsymbol{t}$ to a pseudo-time $\mathbf{t}$ by means of the formula $\boldsymbol{t} = \sqrt{\omega}\, \mathbf{t}$ and get

$$\boldsymbol{r}'' = \epsilon[\mathbf{m}_2 \boldsymbol{r} \boldsymbol{\theta}'^2 + \omega(\mathbf{m}_2 - \mathbf{m}_1) - \tfrac{1}{2}\, \mathbf{m}_2 \omega \boldsymbol{\theta}^2] \tag{9}$$

$$\boldsymbol{r}\boldsymbol{\theta}'' + \omega\boldsymbol{\theta} = -2\boldsymbol{r}'\boldsymbol{\theta}' \tag{10}$$

where the $'$ indicates differentiation with respect to $\mathbf{t}$. Assuming that $\boldsymbol{r}, \boldsymbol{\theta}, \omega$ and $\mathbf{m}_1$ are analytic functions of ϵ:

$$\boldsymbol{r} = r_0 + \epsilon r_1 + \epsilon^2 r_2 + \ldots$$

$$\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \epsilon\boldsymbol{\theta}_1 + \epsilon^2\boldsymbol{\theta}_2 + \ldots$$

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots$$

$$\mathbf{m}_1 = \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \ldots$$

We introduce these expansions into Equations (9) and (10) and collect terms in like powers of ϵ. Since ϵ is an arbitrary parameter, each coefficient of the powers of ϵ must be zero. This results in setting up an infinite system of pairs of second order differential equations which can be solved successively and recursively. The first set of pairs is obtained by setting $\epsilon = 0$. The two equations obtained are:

$$r_0'' = 0 \tag{11}$$

$$r_0\boldsymbol{\theta}_0'' + \omega_0\boldsymbol{\theta}_0 = -2r_0'\boldsymbol{\theta}_0' \tag{12}$$

The solution of Equation (11) is $\mathbf{r}_0 = r_0$, a constant, since we assume as an initial condition that $\mathbf{r}_0'(0) = 0$. The solution of Equation (12) then becomes

$$\boldsymbol{\theta}_0 = \theta_0\cos(kt) \tag{13}$$

where θ_0 is the initial angular displacement, $k^2 = \omega_0/r_0$ and we have assumed that $\theta_0'(0) = 0$. It will turn out that ω_0 plays no role in the final form of the formulas for $\rho(t)$ and $\theta(t)$. In terms of the physical variables, $k^2 = g/\rho(0)$.

If Equation (9) is differentiated with respect to the parameter ϵ and then ϵ is set equal to zero, we obtain:

$$r_1'' = m_2 r_0 \theta_0'^2 - \mu_0\omega_0 + m_2\omega_0(1 - \tfrac{1}{2}\theta_0^2) \tag{14}$$

When the values of θ_0 and θ_0', as given from Equation (13) and its derivative, are substituted into Equation (14), and remembering that $r_0 = r_0$, a constant, then, after some algebraic manipulation, Equation (14) may be written as:

$$r_1'' = \omega_0[-\mu_0 + m_2 + (\tfrac{1}{4})m_2\theta_0^2 - (\tfrac{3}{4})m_2\theta_0^2\cos(2kt)] \tag{15}$$

Since we are seeking a <u>periodic</u> solution for r, the <u>secular term</u> in the general solution of Equation (15), the term that would give rise to a quadratic increase or decrease with time in r_1, will be eliminated if we set:

$$\mu_0 = m_2 + (\tfrac{1}{4})m_2\theta_0^2 \tag{16}$$

Otherwise, r_1 will contain a term of the form $(-\mu_0 + m_2 + (\frac{1}{4})m_2\theta_0^2)t^2/2$ and, depending on the relative magnitude chosen for μ_0 and $(m_2 + (\frac{1}{4})m_2\theta_0^2)$, r_1 increases or decreases quadratically with time. The solution of Equation (15), for the initial conditions $r_1(0) = r_1'(0) = 0$, is:

$$r_1 = (\tfrac{3}{16})r_0 m_2\theta_0^2(\cos(2kt) - 1) \tag{17}$$

When Equation (10) is differentiated with respect to ϵ and then ϵ is set equal to zero, we obtain:

$$r_0\theta_1'' + \omega_0\theta_1 = -r_1\theta_0'' - \omega_1\theta_0 - 2r_1'\theta_0' \tag{18}$$

Since r_0, θ_0 and r_1 are known, we substitute their respective values into the right-hand side of Equation (18) to obtain:

$$r_0\theta_1'' + \omega_0\theta_1 = \theta_0\{-[\omega_1 + (\tfrac{15}{32})m_2\theta_0^2\omega_0]\cos(kt) +$$

$$[(\tfrac{15}{32})m_2\theta_0^2\omega_0]\cos(3kt)\} \tag{19}$$

We are interested in obtaining the particular solution of Equation (19) that remains bounded with increasing time, i.e. we are interested in excluding the secular term. That part of the particular solution of Equation (19) that arises from the term

$$\theta_0[\omega_1 + (\tfrac{15}{32})m_2\, \theta_0{}^2\omega_0]\cos(kt)$$

is of the form $\alpha t\sin(kt)$, where α is a constant depending on the several constants in the equation, namely, r_0, ω_0, ω_1, θ_0, m_2, and k and whose explicit dependency on these is not important. What is important is that unless $\alpha = 0$, the function t sin(kt) oscillates wildly and it is this oscillation that must be eliminated. This is done by choosing

$$\omega_1 = -(\tfrac{15}{32})m_1\theta_0{}^2\omega_0 \tag{20}$$

With this done, the particular solution of Equation (19), with initial conditions $\theta_1(0) = \theta_1{}'(0) = 0$, is:

$$\theta_1 = (\tfrac{15}{256})\theta_0(m_2\theta_0{}^2)[\cos(kt) - \cos(3kt)] \tag{21}$$

Note that if ω_1 is not chosen as in Equation (20), then the solution for θ_1 will oscillate wildly with time. We interpret this by saying that the mathematical procedure fails. Thus, in order to retain mathematical viability, we must pick ω_1 as given in Equation (20).

If we were to proceed no further, we would have the following approximate solutions to Equations (3) and (4) (after setting $\epsilon = 1$):

$$\boldsymbol{r} = r_0[1 + (\tfrac{3}{16})m_2\theta_0{}^2(\cos(2kt) - 1)] \tag{22}$$

$$\boldsymbol{\theta} = \theta_0\{\cos(kt) + (\tfrac{15}{256})(m_2\theta_0{}^2)[\cos(kt) - \cos(3kt)]\} \tag{23}$$

where

$$m_1 = m_2 + (\tfrac{1}{4})m_2\theta_0{}^2 \tag{24}$$

$$\boldsymbol{\omega} = \omega_0[1 - (\tfrac{15}{32})m_2\theta_0{}^2] \tag{25}$$

$$k^2 = \omega_0/r_0 \tag{26}$$

We proceed to get the next higher order terms in the expansions for $\boldsymbol{r}$, $\boldsymbol{\theta}$, m_1, and $\boldsymbol{\omega}$.

By equating the coefficients of ϵ^2 from both sides of Equation (9), we get

$$r_2'' = -[\omega_0\mu_1 + \omega_1(\mu_0 - m_2)] + m_2[r_1\theta_0'^2 + 2r_0\theta_0'\theta_1']$$

$$- \frac{1}{2} m_2[\omega_1\theta_0^2 + 2\omega_0\theta_0\theta_1] \tag{27}$$

After substituting the several quantities already found for their respective values as given in the right–hand side of Equation (27), it may be rewritten as:

$$r_2'' = (\frac{9}{512})m_2^2\theta_0^4\omega_0[4\cos(2kt) + 9\cos(4kt)] \tag{28}$$

and where, in order to preclude the introduction of a secular term in r_2, we had set:

$$\mu_1 = (\frac{63}{512})m_2^2\theta_0^4 \tag{29}$$

The solution of Equation (28), subject to the initial conditions $r_2(0) = r_2'(0) = 0$, is:

$$r_2 = \left[\frac{9}{2^{13}}\right](m_2\theta_0^2)^2 r_0[25 - 16\cos(2kt) - 9\cos(4kt)] \tag{30}$$

By equating the coefficients of ϵ^2 from both sides of Equation (10), we get:

$$r_0\theta_2'' + \omega_0\theta_2 + [r_1\theta_1'' + \omega_1\theta_1 + 2r_1'\theta_1'] +$$

$$[r_2\theta_0'' + \omega_2\theta_0 + 2r_2'\theta_0'] = 0 \tag{31}$$

After substituting the values already found for $r_0,\theta_0,r_1,\theta_1,\omega_0,\omega_1$ and their derivatives where appropriate, we get:

$$r_0\theta_2'' + \omega_0\theta_2 = \left[\frac{9}{2^{14}}\right]m_2^2\theta_0^5\omega_0[163\cos(3kt) - 291\cos(5kt)] \tag{32}$$

and where, in order to preclude the introduction of a secular term in θ_2, we have set:

$$\omega_2 = (\frac{9}{128})m_2^2\theta_0^4\omega_0 \tag{33}$$

The value of θ_2, subject to the initial conditions $\theta_2(0) = \theta_2'(0) = 0$, found by integrating Equation (32) is:

$$\theta_2 = \left[\frac{9}{2^{17}}\right]m_2^2\theta_0^5[66\cos(kt) - 163\cos(3kt) + 97\cos(5kt)] \tag{34}$$

Summarizing the results obtained thus far, we see that after setting $\epsilon = 1$, we get the following approximate solutions for $r(t)$ and $\theta(t)$:

$$r(t) = r_0[1 + (\tfrac{3}{16})m_2\theta_2{}^2(\cos(2kt) - 1) +$$

$$\left[\frac{9}{2^{13}}\right]m_2{}^2\theta_0{}^4(25 - 16\cos(2kt) - 9\cos(4kt)] \tag{35}$$

$$\theta(t) = \theta_0\{\cos(kt) + (\tfrac{15}{256})m_2\theta_0{}^2[\cos(kt) - \cos(3kt)] +$$

$$\left[\frac{9}{2^{17}}\right]m_2{}^2\theta_0{}^4[66\cos(kt) - 163\cos(3kt) + 97\cos(5kt)]\}, \tag{36}$$

where

$$m_1 = m_2 + (\tfrac{1}{4})m_2\theta_0{}^2 + (\tfrac{63}{512})m_2{}^2\theta_0{}^4 \tag{37}$$

$$\omega = \omega_0[1 - (\tfrac{15}{32})m_2\theta_0{}^2 + (\tfrac{9}{128})m_2{}^2\theta_0{}^4] \tag{38}$$

Since $t = (\omega)^{-\frac{1}{2}}\, t$, $r(t) \Rightarrow r(\omega^{-\frac{1}{2}}\, t)$, $\theta(t) \Rightarrow \theta(\omega^{-\frac{1}{2}}\, t)$, i.e. in Formulas (35) and (36) replace t by $\omega^{-\frac{1}{2}}\, t$. Finally,

$$\rho(t) = g\,r(\omega^{-\frac{1}{2}}\, t)$$

and

$$\theta(t) = \theta(\omega^{-\frac{1}{2}}\, t)$$

II.2.b Atwood's Pendululm—non-periodic motion. The configuration, symbols and references to equations are as in the previous section, II.2.a. Whereas in the previous section we used $1 - \theta^2/2$ as the small angle approximation for $\cos\theta$, we shall in this section set $\cos\theta = 1$. Equation (3) remains unchanged but Equation (4) simplifies to:

$$(m_1 + m_2)\,\ddot{r} - m_2 r\dot{\theta}^2 + (m_1 - m_2) = 0 \tag{4.1}$$

Having seen that for the periodic solution to exist, we must have $m_1 > m_2$ (to counteract the centrifugal force produced by the swinging motion of the mass m_2), it is reasonable to expect that if m_1 is less than the critical value as given (approximately) by Equation (37), then the length ℓ would increase indefinitely (or at least to the extent the physical apparatus would permit), whereas if m_1 is greater than this critical value, then ℓ would approach, and become, zero. These observations were confirmed numerically, and in fact the computations were carried out without assuming the small angle approximations, by integrating Equations (1) and (2) (Mittleman[3] and Zeigler[4]).

The special case for which $m_1 = m_2$ was studied extensively using numerical integration in Mittleman[3] and Zeigler[4]. In [5], Jon Lee discussed, using asymptotic methods, both the short time and the long time behavior of this case and provided a theoretical basis for the numerical experiments reported in [3] and [4].

II.3.a Two Countered–Balanced Dumbbells. The system to be described is depicted in Figure 3. The two masses m_1 and m_2 are connected by a massless rigid rod and suspended by inextensible wires from the two pulleys p_1 and p_2. The wires then run horizontally across the top of the apparatus to the pulleys p_3 and p_4 and then drop down to the two masses m_3 and m_4 which are also connected by a massless rigid rod. If the only forces allowed to act on the two masses m_1 and m_2 are gravity and the tensions in the wires and there are no initial conditions on these two masses to produce motion out of the vertical plane, this dumbbell can move only in the vertical plane. For the dumbbell made up of the two masses m_3 and m_4, initial conditions can be chosen so that the four points p_3, p_4, m_3, and m_4 need not be coplanar and in fact, the dumbbell $m_3 - m_4$ need not lie in the vertical plane containing p_3, p_4. The mathematical description that follows is based on these assumptions.

The kinetic energy of the four masses is:

$$T = \frac{1}{2} \sum_{i=1}^{4} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \tag{1}$$

The potential energy of the four masses is:

$$V = \sum_{i=1}^{4} m_i g z_i \tag{2}$$

We introduce the following vector notation: for i = 1,2,3,4 (where appropriate)

$$\vec{r}_i = (x_i, y_i, z_i)$$

$$\vec{p}_i = (p_{ix}, p_{iy}, p_{iz})$$

$$\vec{v}_i = \dot{\vec{r}}_i$$

$$\vec{l}_i = \vec{r}_i - \vec{p}_i$$

$$l_i = [\vec{l}_i \cdot \vec{l}_i]^{\frac{1}{2}}$$

$$\vec{R}_{12} = \vec{r}_1 - \vec{r}_2$$

$$\vec{R}_{34} = \vec{r}_3 - \vec{r}_4$$

$$\vec{g} = (0,0,g)$$

The equations for the constraints are:

$$\phi_1: \qquad (\vec{r}_1 - \vec{r}_2).(\vec{r}_1 - \vec{r}_2) = \vec{R}_{12} \cdot \vec{R}_{12}$$

$$\phi_2: \qquad (\vec{r}_3 - \vec{r}_4).(\vec{r}_3 - \vec{r}_4) = \vec{R}_{34} \cdot \vec{R}_{34}$$

$$\phi_3: \qquad [\vec{l}_1 \cdot \vec{l}_1]^{\frac{1}{2}} + [\vec{l}_3 \cdot \vec{l}_3]^{\frac{1}{2}} =$$

$$[\vec{r}_1 - \vec{p}_1) \cdot (\vec{r}_1 - \vec{p}_1)]^{\frac{1}{2}} + [(\vec{r}_3 - \vec{p}_3)\ (\vec{r}_3 - \vec{p}_3)]^{\frac{1}{2}} = L_1 - d_{13}$$

$$\phi_4: \qquad [\vec{l}_2 \cdot \vec{l}_2]^{\frac{1}{2}} + [\vec{l}_4 \cdot \vec{l}_4]^{\frac{1}{2}} =$$

$$[\vec{r}_2 - \vec{p}_2) \cdot (\vec{r}_2 - \vec{p}_2)]^{\frac{1}{2}} + [(\vec{r}_4 - \vec{p}_4)\ (\vec{r}_4 - \vec{p}_4)]^{\frac{1}{2}} = L_2 - d_{24}$$

where L_1 and L_2 are the lengths of the two connecting wires.

Hamilton's Principle is:

$$\delta \int_{t_0}^{t_1} F(t;x_1,x_2,\ldots,\dot{z}_2,\dot{z}_3)dt = 0$$

and where:

$$F = T - V + \sum_1^4 \lambda_i(t)\phi_i(x_1,x_2,\ldots,z_3,z_4)$$

and where the $\lambda_i(t)$ are, for the moment, Lagrangian parameters. Carrying out the details of the computation leads to the following system of equations:

$$m_1\ddot{\vec{r}}_1 = -m_1\vec{g} + 2\lambda_1(t)(\vec{r}_1 - \vec{r}_2) + \lambda_3(t)\,\frac{\vec{r}_1 - \vec{p}_1}{l_1}$$

$$m_2\ddot{\vec{r}}_2 = -m_2\vec{g} - 2\lambda_1(t)(\vec{r}_1 - \vec{r}_2) + \lambda_3(t)\,\frac{\vec{r}_2 - \vec{p}_2}{l_2}$$

$$m_3\ddot{\vec{r}}_3 = -m_3\vec{g} + 2\lambda_2(t)(\vec{r}_3 - \vec{r}_4) + \lambda_3(t)\,\frac{\vec{r}_3 - \vec{p}_3}{l_3}$$

$$m_4\ddot{\vec{r}}_4 = -m_4\vec{g} - 2\lambda_2(t)(\vec{r}_3 - \vec{r}_4) + \lambda_4(t)\,\frac{\vec{r}_4 - \vec{p}_4}{l_4}$$

to which we must adjoin the four constraint equations.

This second order system of differential equations is replaced by an equivalent system of first order equations so as to facilitate using numerical integration; in this case we chose Runga–Kutta–Fehlberg[6].

$$\dot{\vec{r}}_1 = \vec{v}_1$$

$$m_1\dot{\vec{v}}_1 = -m_1\vec{g} + 2\lambda_1(t)(\vec{r}_1 - \vec{r}_2) + \lambda_3(t)\,\frac{\vec{r}_1 - \vec{p}_1}{l_1}$$

$$\dot{\vec{r}}_2 = \vec{v}_2$$

$$m_2\dot{\vec{v}}_2 = -m_2\vec{g} - 2\lambda_1(t)(\vec{r}_1 - \vec{r}_2) + \lambda_2(t)\,\frac{\vec{r}_2 - \vec{p}_2}{l_2}$$

$$\dot{\vec{r}}_3 = \vec{v}_3$$

$$m_3\dot{\vec{v}}_3 = -m_3\vec{g} + 2\lambda_2(t)(\vec{r}_3 - \vec{r}_4) + \lambda_3(t)\,\frac{\vec{r}_3 - \vec{p}_3}{l_3}$$

$$\dot{\vec{r}}_4 = \vec{v}_4$$

$$m_4\dot{\vec{v}}_4 = -m_4\vec{g} - 2\lambda_2(t)(\vec{r}_3 - \vec{r}_4) + \lambda_4(t)\,\frac{\vec{r}_4 - \vec{p}_4}{l_4}$$

and again we must remember to adjoin the four restraint equations.

These equations are not as yet in a form suitable for numerical integration since we do not know how to handle the four $\lambda_i(t)$. We proceed as follows. With time, t, as the independent variable, differentiate each of the four constraint equations, <u>twice</u>. By adjoining these eight equations to the twenty-four that contain $\vec{\dot{r}}_i$ and $\vec{\dot{v}}_i$, we can eliminate all time derivatives. The result of these lengthy and tedious algebraic manipulations is a <u>linear</u> system of four equations in the four $\lambda_i(t)$. The numerical integration can now begin.

Pick initial conditions for $r_i(0)$ and $v_i(0)$ consistent with the constraint equations. Although it is possible to parameterize these four constraint equations, this is not recommended. Instead, use a Newton-Raphson procedure. With these values, calculate the $\lambda_i(0)$. Enter into the Runga-Kutta-Fehlberg routine. As the program progresses through the several steps of the R-K-F routine, it will be necessary to know (i.e. calculate) intermediate values of the $\vec{r}_i$ and $\vec{v}_i$; these will be got from the constraint equations. Thus, even for one step in the R-K-F algorithm, to get $\vec{r}_i(t_1)$ and $\vec{v}_i(t_1)$, it is necessary to make several intermediate computations using the constraint equations. The number of such intermediate excursions will depend on which order R-K-F you chose to use. The details of the code are not included in this report; the programming and running of it was done by Arnel Pacia of WRDC/FIBG and would be available from that source.

While there is literature on systems of differential equations with algebraic constraints, none seemed immediately applicable to this particular problem and the method described, therefore, was invented for this application.

Several computer runs were made using this code and aside from the general oscillatory patterns that we had come to expect there was one surprise. When we picked equal lengths for the two wires connecting $m_1 - m_3$ and $m_2 - m_4$ and displaced the $m_3 - m_4$ dumbbell from its rest position and released it so that it would swing, we calculated the distance from the center of the $m_3 - m_4$ dumbbell to the center of the $m_1 - m_2$ dumbbell measuring from the first center over the top of the apparatus and then down to the other center, we found that this distance changed and depended on the angular orientations of the two dumbbells. It was always less, albeit the amount was very small, than the common lengths of the connecting wires. While this is of no significance in this problem, it portends difficulties if one were to try to formulate a problem for three masses in a row, again rigidly connected, counter-balanced by a similar arrangement and connected by three wires of equal length. It seems to this writer that the problem is akin to the problem of supporting a rod at three or more points; this problem is known to be statically indeterminate and elastic theory is required for its solution.

II.3.b Two Counter-Balanced Dumbbells Connected By Three Wires. From the discussion presented in the previous paragraph, if three wires are used to join corresponding points on the two dumbbells, and one of the dumbbells is set into motion, there could be no tension on the center wire. This writer would infer then that the problem of three connecting wires will yield results not different from those obtained from the problem with two connecting wires. The evidence is as indicated above but a more convincing argument may be required.

II.4 One Dumbbell Counter–Balanced By Weights. Suppose the configuration discussed in II.3.a is modified by eliminating the constraint ϕ_1, that is, by assuming that the masses m_1 and m_2 are not rigidly connected, in fact, are not connected at all. Under this assumption the only motion possible for these two masses is vertical, i.e. $\dot{x}_1(t) = \dot{y}_1(t) = \dot{x}_2(t) = \dot{y}_2(t) = 0$. The equations of motion for this situation are trivially derivable from those given above by paralleling the derivation without the ϕ_1 constraint. We have not carried out the details and thus have no numerical results to report.

As yet, another possibility would be to initially have m_3, m_4 and the rod connecting them at rest and impart initial velocities to m_1 and m_2. The above equations certainly suffice; we have not, however, carried out the details and have no results to report.

It would be reasonable now to counter-balance the dumbbell by three or more weights and study the motion. Obviously this modification circumvents the problem of static indeterminance and might provide further insight into the motion. This has not been done.

II.5.a Counter–Balanced Rigid Rod — With Two Wires. We start with a rod of length l, uniform density ρ_0, and uniform cross–section A_0. The center of mass of the rod is at the point 0. Consider this rod in an inertial coordinate system (x,y,z) such that one end of the rod is at the point $\vec{r}_1 = (x_1,y_1,z_1)$ and the other end is at the point $\vec{r}_2 = (x_2,y_2,z_2)$. If $\cos\alpha_1$, $\cos\alpha_2$, $\cos\alpha_3$ are the direction cosines of the rod with respect to the inertial coordinate system, then $(\cos\alpha_1, \cos\alpha_2, \cos\alpha_3) = (\vec{r}_2 - \vec{r}_1)/l$. The coordinates of an arbitrary point on the rod is given by the vector

$$\vec{r} = \vec{r}_1 + (\vec{r}_2 - \vec{r}_1)\sigma \qquad 0 \le \sigma \le 1$$

In the rod, set up a right–hand coordinate system (x′,y′,z′), with the x′ direction along the length of the rod, the y′ and the z′ directions both orthogonal to x′ and to each other, but otherwise unspecified. The origin of this system is taken at 0. If the linear mass density of the rod is given by $\rho(x') \ge 0$ (in our case, $\rho(x') = \rho_0$), then the mass m_1 of the rod is given by

$$m_1 = \int \rho(x')\, dx' = \rho_0 l$$

and

$$0 = \int \rho(x')\, x'\, dx'$$

and

$$I_1 = \int \rho(x')(x')^2 dx' = (m_1 l^2)/12$$

and where the integration is on the interval $\left[-\frac{l}{2}, \frac{l}{2}\right]$. Let the origin of the (x', y', z') system have coordinates (x_{1c}, y_{1c}, z_{1c}) in the inertial system. Then an arbitrary point P in space may be referenced in both systems; the relation between coordinate values in the two systems being given by

$$x = x_{1c} + a_{11}x' + a_{12}y' + a_{13}z'$$

$$y = y_{1c} + a_{21}x' + a_{22}y' + a_{23}z'$$

$$z = z_{1c} + a_{31}x' + a_{32}y' + a_{33}z'$$

and where $A = (a_{ij})$ is an orthogonal matrix. For points on the rod, $y' = z' = 0$, and we have

$$x = x_{1c} + a_{11}x'$$

$$y = y_{1c} + a_{21}x'$$

$$z = z_{1c} + a_{31}x'$$

and, in terms of the angles previously defined:

$$a_{11} = \cos \alpha_1$$

$$a_{21} = \cos \alpha_2$$

$$a_{31} = \cos \alpha_3$$

To calculate the kinetic energy of the rod, we consider first an element of length dx'. The mass dm associated with this element is $\rho(x')dx' = \rho_0 dx'$ and the square of the velocity, v^2, is:

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

and where

$$\dot{x} = \dot{x}_{1c} + \dot{a}_{11}x'$$

$$\dot{y} = \dot{y}_{1c} + \dot{a}_{21}x'$$

$$\dot{z} = \dot{z}_{1c} + \dot{a}_{31}x'$$

The kinetic energy of this element is equal to $(\frac{1}{2})[\rho(x')dx']v^2$ and after summing over the length of the rod, the kinetic energy is

$$T_1 = \tfrac{1}{2}(\dot{x}_{1c}^{\,2} + \dot{y}_{1c}^{\,2} + \dot{z}_{1c}^{\,2})\, m_1 + \tfrac{1}{2}(\dot{a}_{11}^{\,2} + \dot{a}_{21}^{\,2} + \dot{a}_{31}^{\,2})\, I_1$$

This is nothing more than the well known result that the total kinetic energy of a rigid body is equal to the translational kinetic energy of the body as if all its mass were concentrated at the center of mass and moving with the velocity of the center plus the kinetic energy of rotation about the center of mass.

The three variables a_{11}, a_{21}, a_{31} are not independent but are related by the equation $a_{11}^{\,2}+a_{21}^{\,2}+a_{31}^{\,2} = 1$. Independent variables θ_1, ϕ_1 may be introduced as follows: let $a_{31} = \cos\theta_1$, $a_{11} = \cos\phi_1 \sin\theta_1$, $a_{21} = \sin\phi_1 \sin\theta_1$. Then

$$\dot{a}_{11}^{\,2} + \dot{a}_{21}^{\,2} + \dot{a}_{31}^{\,2} = \dot{\theta}_1^{\,2} + \sin^2\theta\, \dot{\phi}_1^{\,2}$$

The potential energy of the rod is:

$$V_1 = \int \rho(x')\, z\, g dx' = \int \rho(x')(z_{1c} + a_{31}x')\, g dx'$$

$$= z_{1c} g \int \rho(x')\, dx' + a_{31} g \int \rho(x') x' dx' = m_1 g\, z_{1c}$$

The Lagrangian function for the rod is:

$$L_1 = T_1 - V_1$$

Since a second rod is to be included in the configuration being studied, we would repeat the above computation with all subscripts 1 replaced by subscripts 2. This would lead to the following values for T_2, V_2 and L_2.

$$T_2 = \tfrac{1}{2}(\dot{x}_{2c}^{\,2} + \dot{y}_{2c}^{\,2} + \dot{z}_{2c}^{\,2})m_2 + \tfrac{1}{2}(\dot{b}_{11}^{\,2} + \dot{b}_{21}^{\,2} + \dot{b}_{31}^{\,2})\, I_2$$

and the direction cosines (a_{ij}) have been replaced by the corresponding direction cosines for the second rod (b_{ij}). The potential energy is given by:

$$V_2 = m_2 g z_{2c}$$

and the Lagrangian function $L_2 = T_2 - V_2$. There remain the constraints imposed by the wires connecting the two rods.

We develop the case for two wires connecting the two rods. Refer to Figure 4. The positions of the four pulleys in the inertial system are given by $\vec{p}_i = (p_{1i}, p_{2i}, p_{3i})$,

$i = 1,2,3,4$. We assume that all four p_{3i} are at the same height above the ground plane, although this is not essential. The distances $|\vec{p}_1 - \vec{p}_3| = |\vec{p}_2 - \vec{p}_4| = d_{13} = d_{24}$ and $|\vec{p}_1 - \vec{p}_2| = |\vec{p}_3 - \vec{p}_4| = d_{12} = d_{34}$ are assumed to be *constant*. We shall refer to the wire passing over pulleys 1 and 3 as wire 1 and the wire passing over pulleys 2 and 4 as 2 and denote the length of wire 1 by l_1 and the length of wire 2 by l_2. Wire 1 is connected to rod 1 at the point $(x_1^*, 0, 0)$ in the coordinate system fixed in the rod 1 and is connected to rod 2 at the point $(x_3^*, 0, 0)$ in the coordinate system fixed in that rod. In the common inertial coordinate system, the coordinates of these two points are:
$(x_{1c} + a_{11}x_1^*, y_{1c} + a_{21}x_1^*, z_{1c} + a_{31}x_1^*)$ and $(x_{2c} + b_{11}x_3^*, y_{2c} + b_{21}x_3^*, z_{2c} + b_{31}x_3^*)$ respectively. We shall refer to this first triplet of values by $\vec{q}_1$ and the second triplet by $\vec{q}_3$. The length l_1 of wire 1 is then given by

$$\Gamma_1: \qquad |\vec{q}_1 - \vec{p}_1| + d_{13} + |\vec{p}_3 - \vec{q}_3| = l_1$$

In strictly analogous fashion, wire 2 is connected to rod 1 at the point $(x_2^*, 0, 0)$ and to rod 2 at the point $(x_4^*, 0, 0)$. In the inertial coordinate system, we would have
$\vec{q}_2 = (x_{1c} + a_{11}x_2^*, y_{1c} + a_{21}x_2^*, z_{1c} + a_{31}x_2^*)$ and
$\vec{q}_4 = (x_{2c} + b_{11}x_4^*, y_{2c} + b_{21}x_4^*, z_{2c} + b_{31}x_4^*)$. The length l_2 of wire 2 is then given by:

$$\Gamma_2: \qquad |\vec{q}_2 - \vec{p}_2| + d_{24} + |\vec{p}_4 - \vec{q}_4| = l_2$$

Note that Γ_1 and Γ_2 represent the constraints on the system. The coordinates of the $\vec{p}_i$ are assumed to be fixed but the coordinates of the $\vec{q}_i$ are variable.

While the time t is the only independent variable, the dependent variables are x_{1c}, y_{1c}, z_{1c}, θ_1, ϕ_1, x_{2c}, y_{2c}, z_{2c}, θ_2, ϕ_2. We are now in a position to form the generalized Lagrangian:

$$L^* = L_1 + L_2 + \lambda_1(t)\,\Gamma_1 + \lambda_2(t)\,\Gamma_2$$

and obtain the Euler–Lagrange equations by taking

$$\delta \int_{t_0}^{t_1} L^* \, dt = 0$$

We leave the computation at this point because it parallels both theoretically and numerically the work presented above. Computer codes were not written for this case; time did not permit.

II.5.b Counter–Balanced Rigid Rods — With More Than Two Wires. If the two wires are attached symmetrically to the rods at the points $(\pm x_1^*, 0, 0)$, and $(\pm x_3^*, 0, 0)$ and rod 2 is set swinging while the initial conditions for rod 1 are chosen so that it can move only in the $[y_1, z_1]$ plane, then, I believe that the distance between the centers of the two rods, as measured over the top of the apparatus, would be less than the distance similarly defined if measured when both rods are at rest. Thus, for example, if a third wire looped over the top of the apparatus were to join the two centers, there would be slack during most of the time that rod 2 is swinging. The situation reduces then to one of static indeterminancy and it is questionable, in my mind that this method of approach, namely two counter–balanced rigid rods connected by three or more wires can be correctly solved by simply adding additional constraints. Recourse would need be made to elastic theory.

If one forgoes the second rod and attaches the one rod to a set of isolated point masses by wires stretched over the top of the apparatus, the problem becomes tractable and the methods described above can be used to analyze and describe the motion. These details have not been carried out.

II.6.a Two Counter–Balanced Flexible Rods — With Two Wires. In this section, we set up the Lagrangian function of two counter–balanced flexible rods. The general appearance of the apparatus is as for the case of two rigid rods; as may be expected, however, further analysis is required.

We start with an inertial coordinate system (x_1, y_1, z_1) and consider a rod of uniform density ρ_0 and cross–section A, whose center of mass is at the point 0: (x_1^0, y_1^0, z_1^0) in the inertial coordinate system. Let (x_2, y_2, z_2) be a coordinate system whose origin is at 0 and whose axes are and will remain parallel to those of the (x_1, y_1, z_1) system. The (x_2, y_2, z_2) axes will move as the center of mass of the rod moves. Let (x_3, y_3, z_3) be a coordinate system whose origin is also a 0 but whose axes coincide with the principal axes of inertia of the rod; the x_3 axis lying along the long axis of the rod. (Figure 5.a.)

Let P be a point in space whose coordinates relative to the (x_3, y_3, z_3) axes are $(\alpha_3, \beta_3, \gamma_3)$ and whose coordinates in the (x_2, y_2, z_2) system are $(\alpha_2, \beta_2, \gamma_2)$, then

$$(\alpha_2, \beta_2, \gamma_2) = A(a_{ij})(\alpha_3, \beta_3, \gamma_3)^T$$

where $A(a_{ij})$ is an orthogonal matrix specifying the orientation of the (x_3, y_3, z_3) system in the (x_2, y_2, z_2) system and $(\alpha_3, \beta_3, \gamma_3)^T$ stands for the transpose of $(\alpha_3, \beta_3, \gamma_3)$.

If the coordinates of the point P in the inertial system are $(\alpha_1,\beta_1,\gamma_1)$, then

$$(\alpha_1,\beta_1,\gamma_1) = (x_1{}^0,y_1{}^0,z_1{}^0) + A(a_{ij})(\alpha_3,\beta_3,\gamma_3)^T$$

We consider now the rod flexed about the y_3 direction. Before bending, a cross–section in a plane perpendicular to the y_3 direction remains, after bending, in a plane perpendicular to the y_3 direction. Such a cross–section, in a plane parallel to the (x_3,z_3) plane, is depicted in Figure 5.b.

Suppose now that the rod, before being bent, had been sectioned into small lengths by a set of planes perpendicular to the x_3 axis. Let us look at one such section and the partitioning planes after bending. Grossly, this is indicated in Figure 5.c; the two planes, which had been parallel before bending, now intersect in a line parallel to the y_3 axis and pass through the point marked **0′** in that figure. An enlarged version of that section is given in Figure 5.d.

If $(\alpha_3,\beta_3,\gamma_3)$ are the coordinates of the center of mass of the element of volume depicted in Figure 5.d, then we set up a fourth coordinate system (x_4,y_4,z_4) whose origin is at $(\alpha_3,\beta_3,\gamma_3)$ and whose axes are parallel to the (x_3,y_3,z_3) axes. In this coordinate system, the mass and volume of the section, which we denote by Δm and Δv, are related by the equation $\Delta m = \rho_0 \Delta v$.

Let $(\alpha_4,\beta_4,\gamma_4)$ be the coordinates of a point **P*** in the element Δv depicted in Figure 5.d. Since the (x_4,y_4,z_4) system is parallel to the $(x_3,y_3 z_3)$ system and $(\alpha_3,\beta_3,\gamma_3)$ are the coordinates of the center of mass of the element Δv in the (x_3,y_3,z_3) system, then the coordinates of **P*** in the inertial system are:

$$(x_1,y_1,z_1) = (x_1{}^0,y_1{}^0,z_1{}^0) + A(a_{ij})(\alpha_3+\alpha_4,\beta_3+\beta_4,\gamma_3+\gamma_4)^T$$

The kinetic energy of this element of mass is:

$$\frac{1}{2}\iiint_{(\Delta v)} \rho_0(\dot{x}_1{}^2+\dot{y}_1{}^2+\dot{z}_1{}^2)\, dx_4 dy_4 dz_4$$

where

$$(\dot{x}_1,\dot{y}_1,\dot{z}_1) = (\dot{x}_1{}^0,\dot{y}_1{}^0,\dot{z}_1{}^0)+A(a_{ij})(\dot{\alpha}_3,\dot{\beta}_3,\dot{\gamma}_3)^T+\dot{A}(a_{ij})(\alpha_3+\alpha_4,\beta_3+\beta_4,\gamma_3+\gamma_4)^T$$

The "phase space" variables are:

$$(\dot{x}_1{}^0,\dot{y}_1{}^0,\dot{z}_1{}^0),\ \dot{A}(a_{ij}),\ (\alpha_3,\beta_3,\gamma_3),\ (\dot{\alpha}_3,\dot{\beta}_3,\dot{\gamma}_3)$$

since the variables $(\alpha_4, \beta_4, \gamma_4)$ are dummy variables in the integration.

To calculate the potential energy of this element of mass, we need:

$$z_1 = z_1^{0} + a_{31}(\alpha_3+\alpha_4) + a_{32}(\beta_3+\beta_4) + a_{33}(\gamma_3+\gamma_4)$$

so that the potential energy is:

$$\underset{(\Delta v)}{\int\int\int} \rho_0 g[z_1^{0} + a_{31}(\alpha_3+\alpha_4) + a_{32}(\beta_3+\beta_4) + a_{33}(\gamma_3+\gamma_4)]\, dx_4 dy_4 dz_4$$

$$= \rho_0 g[z_1^{0} + a_{31}(\alpha_3) + a_{32}(\beta_2) + a_{33}(\gamma_3)]\, \Delta v$$

$$= \Delta m g[z_1^{0} + a_{31}(\alpha_3) + a_{33}(\beta_3) + a_{33}(\gamma_3)]$$

The "phase space" variables are. z_1^{0}, $(\alpha_3, \beta_3, \gamma_3)$, and (a_{31}, a_{32}, a_{33}).

We calculate the strain energy U in the rod as follows: Let (dU/dv), (where dv is an element of volume), be the work of deformation so that $(dU/dv) = \frac{1}{2}\sigma\epsilon$ where σ is the stress in the $x_3(=x_4)$ direction and ϵ the corresponding strain (deformation). From Hooke's law $\sigma = E\epsilon$, where E is Young's modulus, and so the quantity we need to calculate is $\frac{1}{2}\frac{\sigma^2}{E} = \frac{1}{2}E\epsilon^2$. We refer again to Figure 5.d.

Prior to bending, the sections mm and pp were parallel; after bending these two sections intersect in a line (whose trace in Figure 5.d is $0'$) parallel to the y_3 axis. The longitudinal fibres, (those measured along the x_4 axis) undergo extension on the convex side of the neutral axis, whereas those on the concave side are compressed. The arc nq is the trace of the surface in which the fibres are neither compressed nor extended during bending (the neutral surface).

The elongation of a fiber at a distance z_4 above the neutral surface may be found by first drawing qs parallel to mn. We look next at the two triangles stq and nq0′. The radial line 0′qt cuts the two circular arcs at right angles, i.e. we assume that the angles nq0′ and stq are both right angles. Also, since sq is parallel to n0′, the line tqp0′ may be considered a transversal intersecting two parallel lines and thus angle sqt = angle n0′q. The two triangles are similar and their corresponding sides are proportional. The "<u>unit</u>" extension of the fiber rs is $(st/nq) = (qs/0'n) = -(z_4/\rho)$, where ρ = radius of curvature of the arc nq. [Note: do not confuse ρ and ρ_0, the density of the rod; the negative sign comes from the fact that in elementary calculus, the curvature is considered positive if the curve bends convex upward.) Thus, the strains of the longitudinal fibers are proportional to their distances from the neutral surface and inversely proportional to the radius of curvature of the neutral surface. Since we are assuming that $(dz_3/dx_3) << 1$, we are assuming that the curvature may be taken as (d^2z_3/dx_3^2). This permits us to write the unit energy

$$(dU/dv) = \tfrac{1}{2}E[z_4(d^2z_3/dx_3^2)]^2$$

Thus:
$$(\Delta U)_x = \tfrac{1}{2}E(d^2z_3/dx_3^2)^2 \int\int\int z_4^2 dx_4 dy_4 dz_4$$

$$- \tfrac{1}{2}E(d^2z_3/dx_3^2)^2(\Delta x_4) \int\int z_4^2 dy_4 dz_4$$

If we let

$$I = \int\int z_4^2 dy_4 dz_4 ,$$

then

$$(\Delta U)_x = \tfrac{1}{2}E(d^2z_3/dx_3^2)^2(\Delta x_4)\ I$$

$$= \tfrac{1}{2}EI(d^2z_3/dx_3^2)^2(\Delta x_3)$$

Continuing, the kinetic energy of the rod =

$$\int [\tfrac{1}{2}\rho_0 \underset{(\Delta v)}{\int\int\int} (\dot{x}_1^2+\dot{y}_1^2+\dot{z}_1^2)dx_4 dy_4 dz_4]\ dx_3 ,$$

the potential energy of the rod =

$$\int \Delta mg\ [z_1^0 + a_{31}(\alpha_3) + a_{32}(\beta_3) + a_{33}(\gamma_3)]\ dx_3$$

and the strain energy of the rod =

$$\int \tfrac{1}{2}EI(d^2z_3/dx_3^2)^2(dx_3).$$

The integration in the x_3 direction is over the length of the rod, $[-\frac{l}{2}, \frac{l}{2}]$.

This computation must be repeated for the second rod which is to be used to counter-balance the first one. In addition, and paralleling the method used in Section II.5, two constraints Γ_1 and Γ_2 are to be introduced. The Lagrangian function takes the form:

$$L = \text{kinetic energy} - \text{potential energy} - \text{strain energy} + \lambda_1(t)\Gamma_1 + \lambda_2(t)\Gamma_2$$

The Euler–Lagrange equations are then obtained from Hamilton's principle:

$$\delta \int_{t_0}^{t_1} L\, dt = 0$$

Whereas in the previous examples, the Euler–Lagrange equations were ordinary differential equations with algebraic constraints, in this case they are partial differential equations with constraints. Numerical procedures, such as the finite element method, would have to be extended to take into account these constraints.

II.6.b If more than two wires are used to join the two rods, the problem assumes another dimension of complexity. In the previous section, the flexibility was assumed limited to vibrations about the y_3 axis; the bar was treated as if it were rigid with respect to bending in the z_3 direction. Whether a third wire would provide support or simply be slack when the rod is vibrating only about the y_3 direction is not known.

In order to get a better understanding of what is happening, it may be necessary to remove the restriction that the connecting wires be inextensible and allow for elastic connecting wires as was done in Section II.1.c.

The comments of this section are purely speculative. It may be possible to get theoretical answers. Numerical results should be obtainable with reasonable effort.

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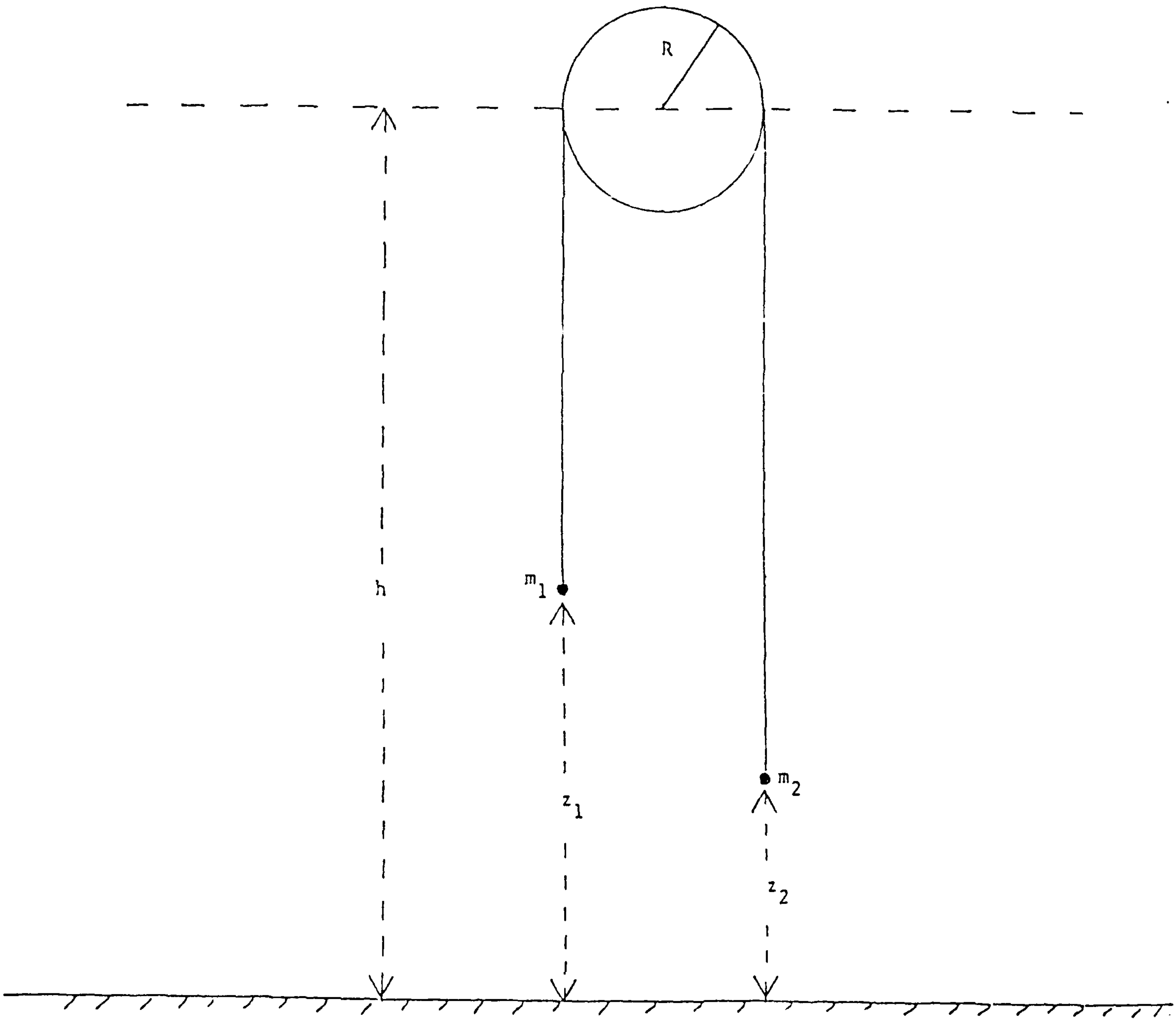


Figure 1.a
The classic Atwood's machine.

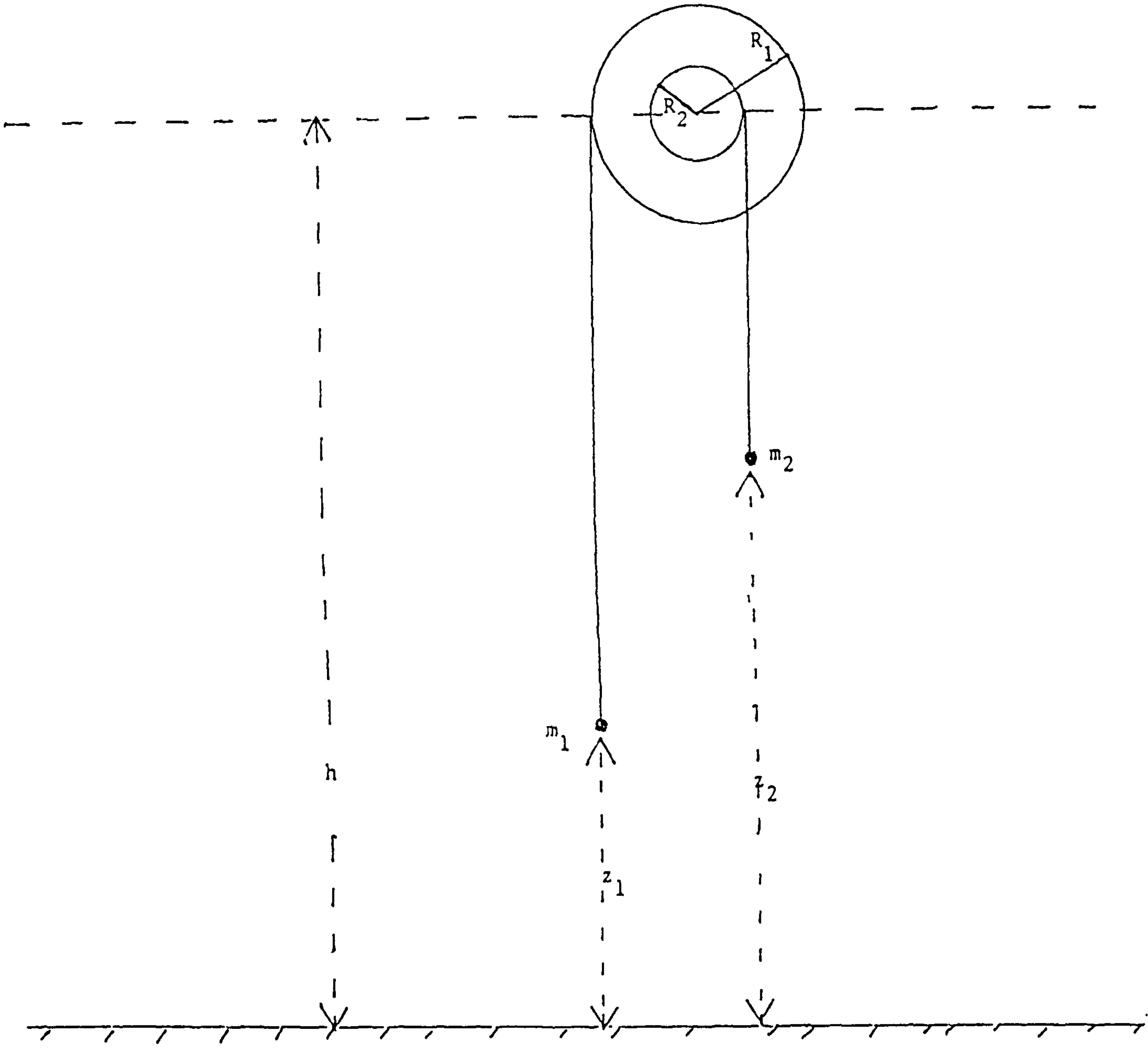


Figure 1.b
An Atwood's machine with unequal radii.

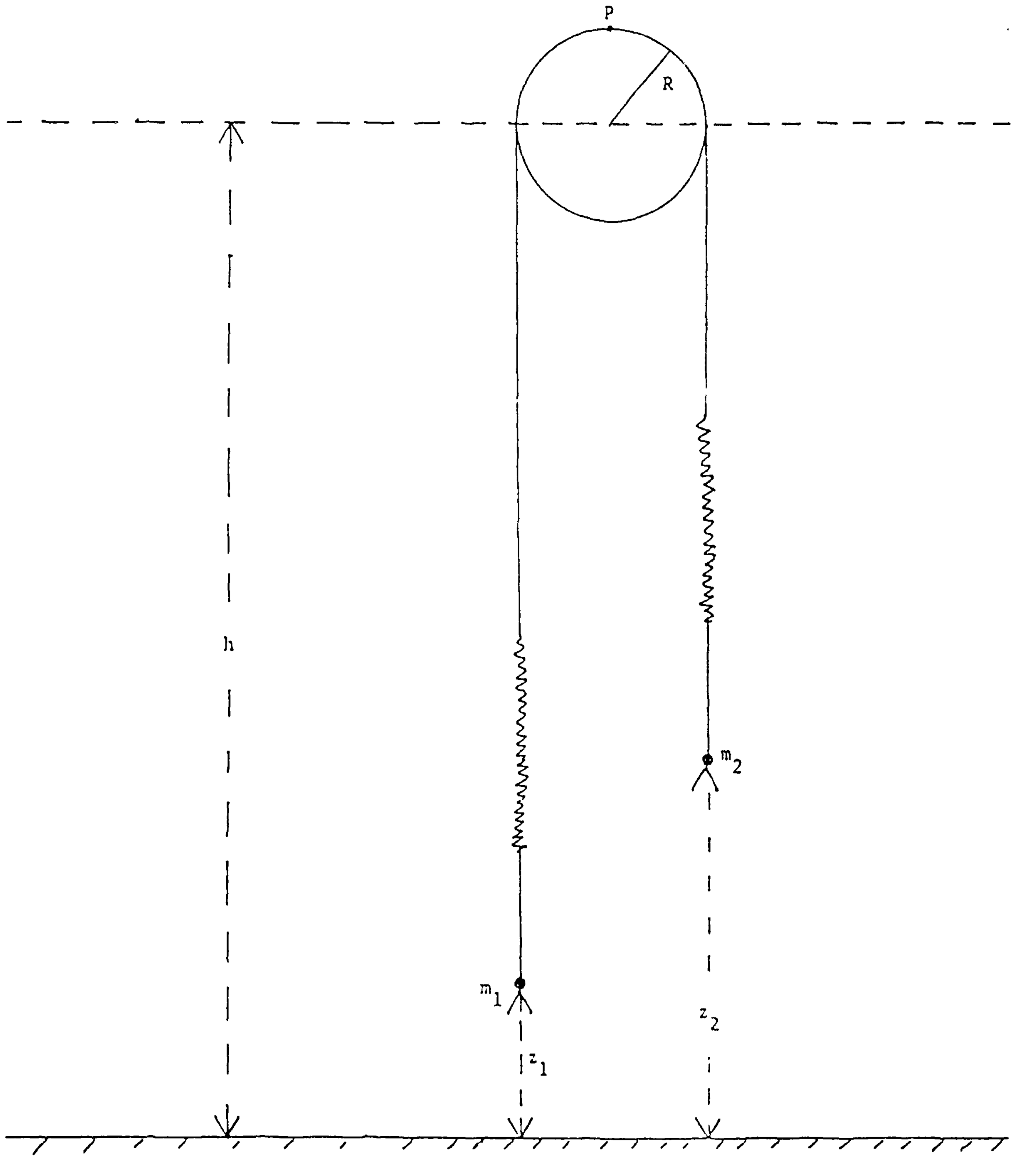


Figure 1.c
An Atwood's machine with elastic wires.

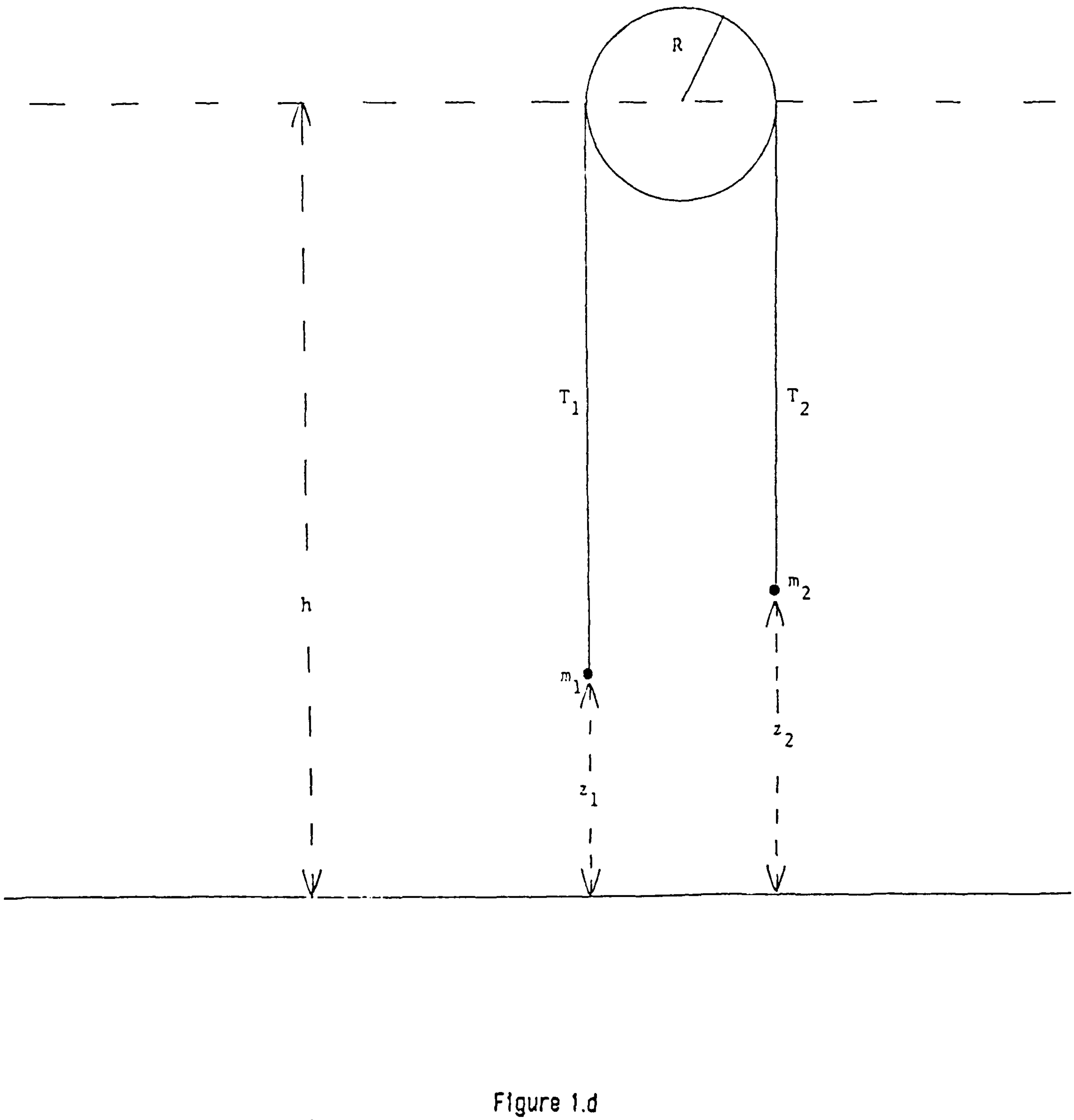


Figure 1.d
An Atwood's machine with friction on the bearing.

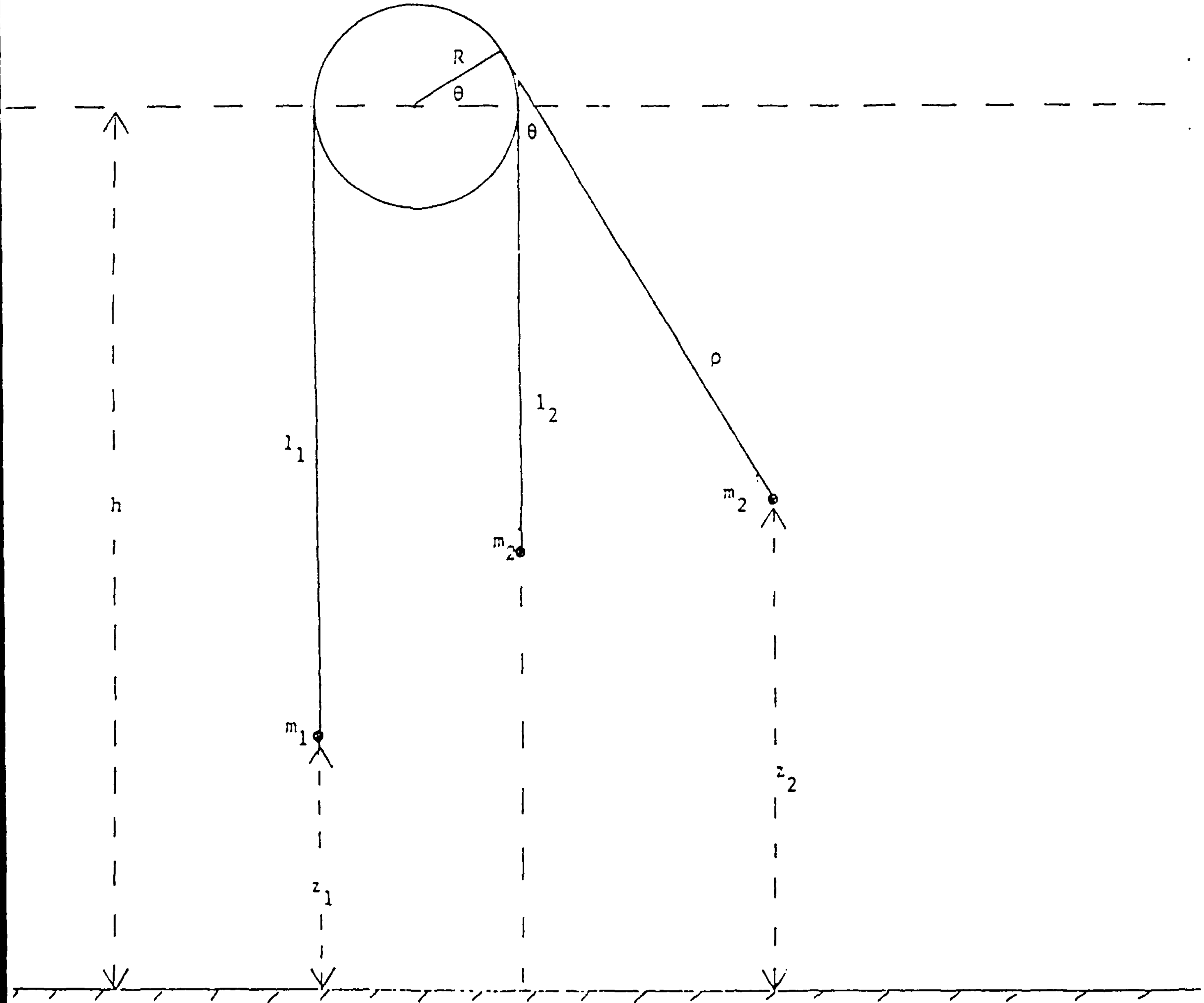


Figure 2
An Atwood's pendulum.

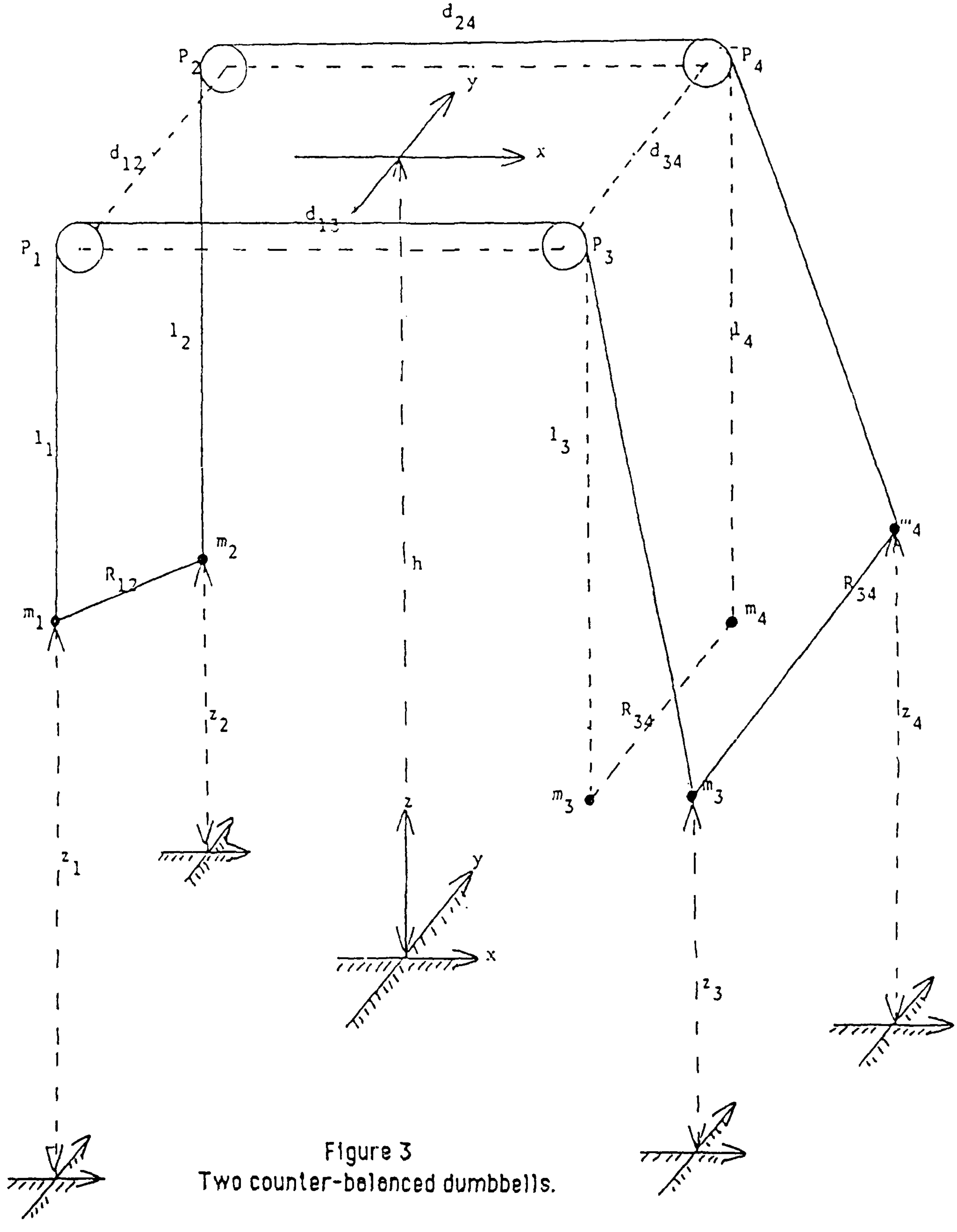


Figure 3
Two counter-balanced dumbbells.

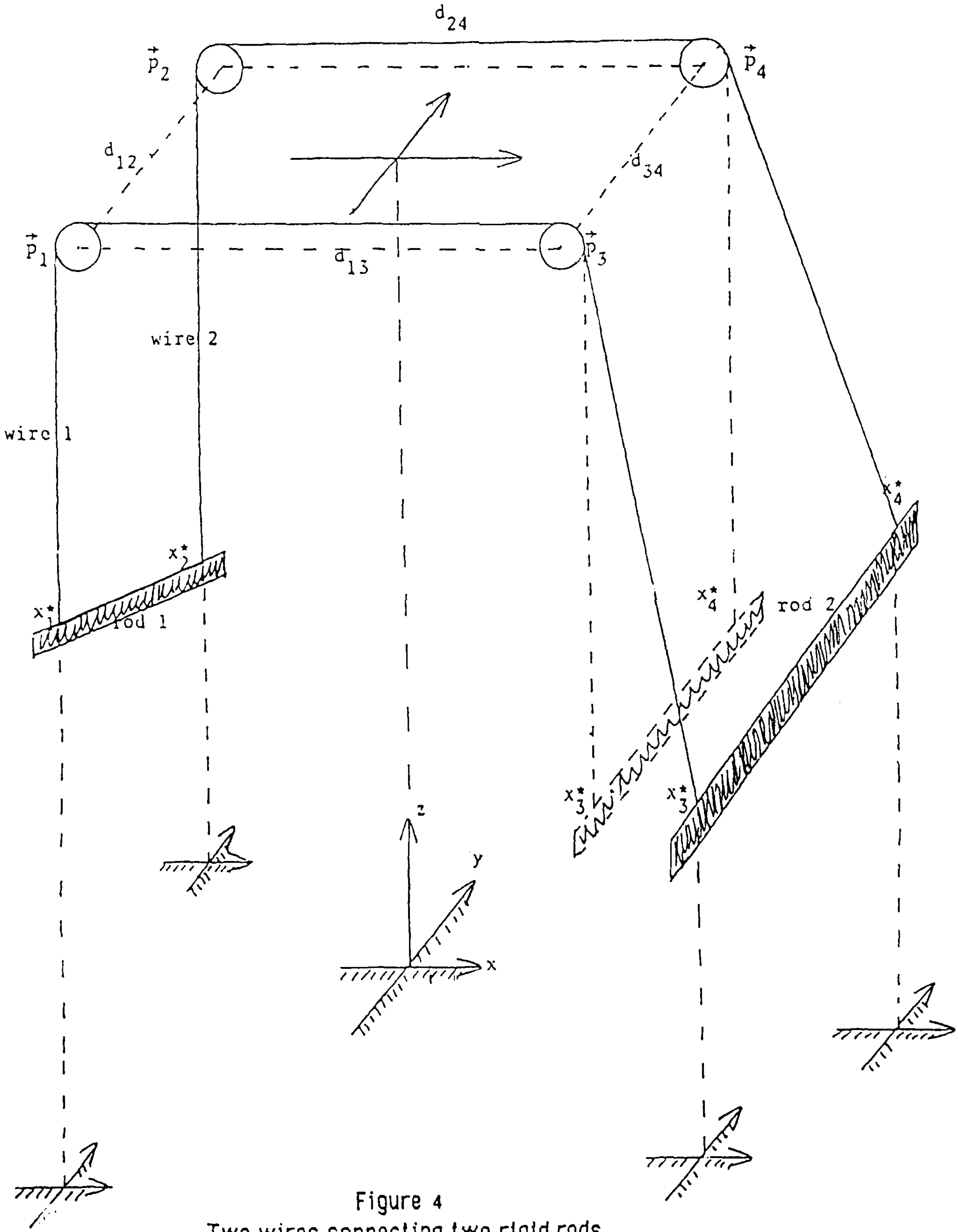
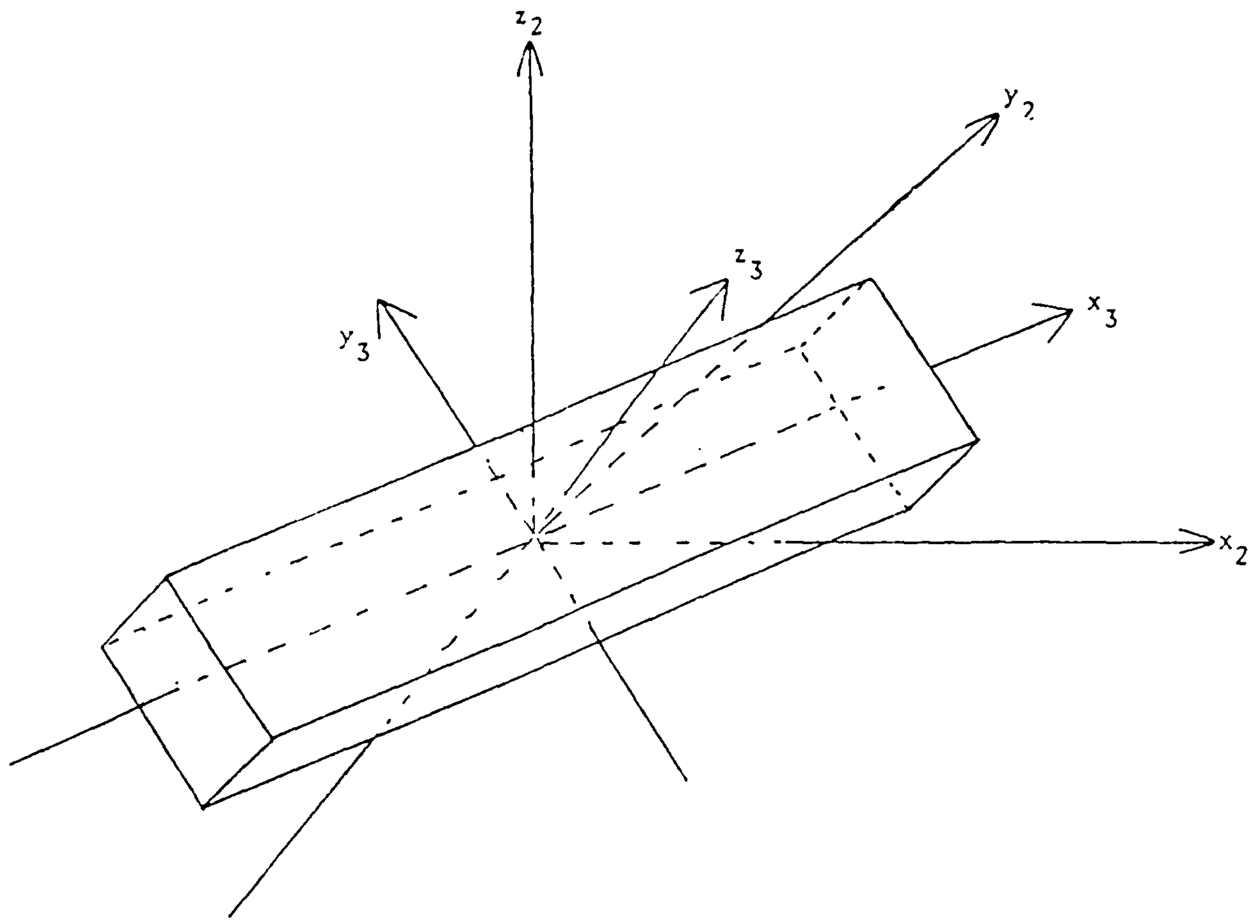


Figure 4
Two wires connecting two rigid rods.



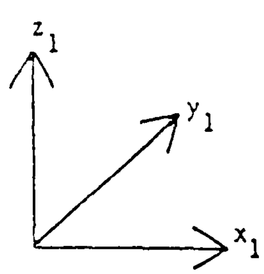


Figure 5.a
Orientation of the unflexed rod in space.

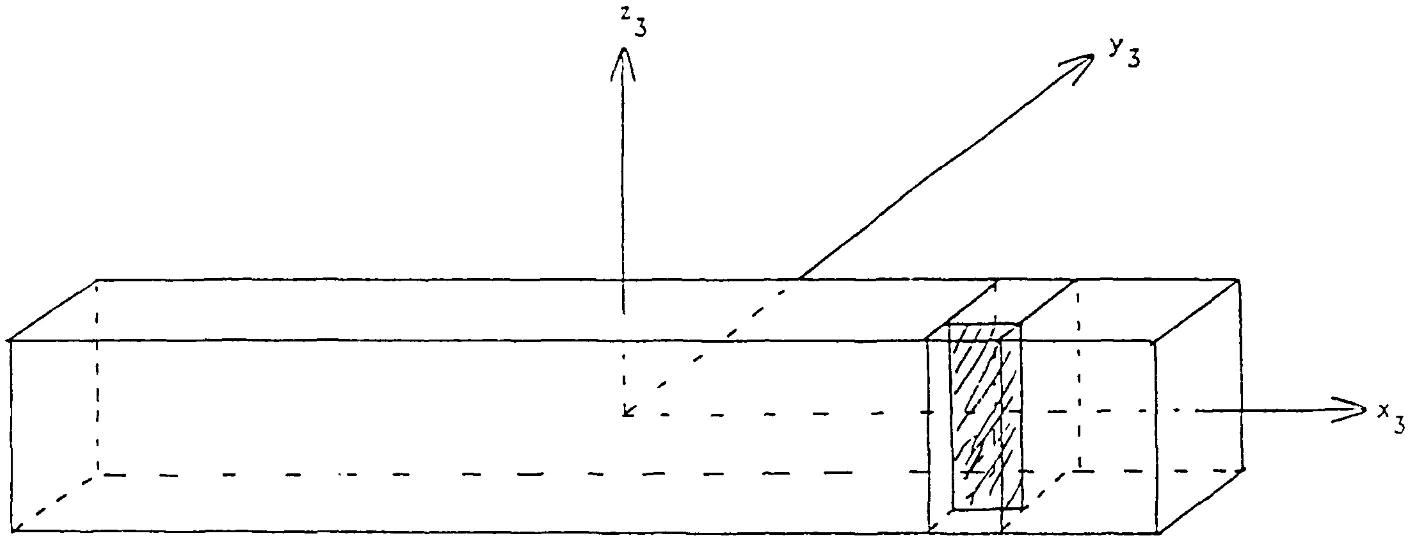


Figure 5.b
A section of a slice of the rod,
a cross-section perpendicular to the y_3 axis,
before bending

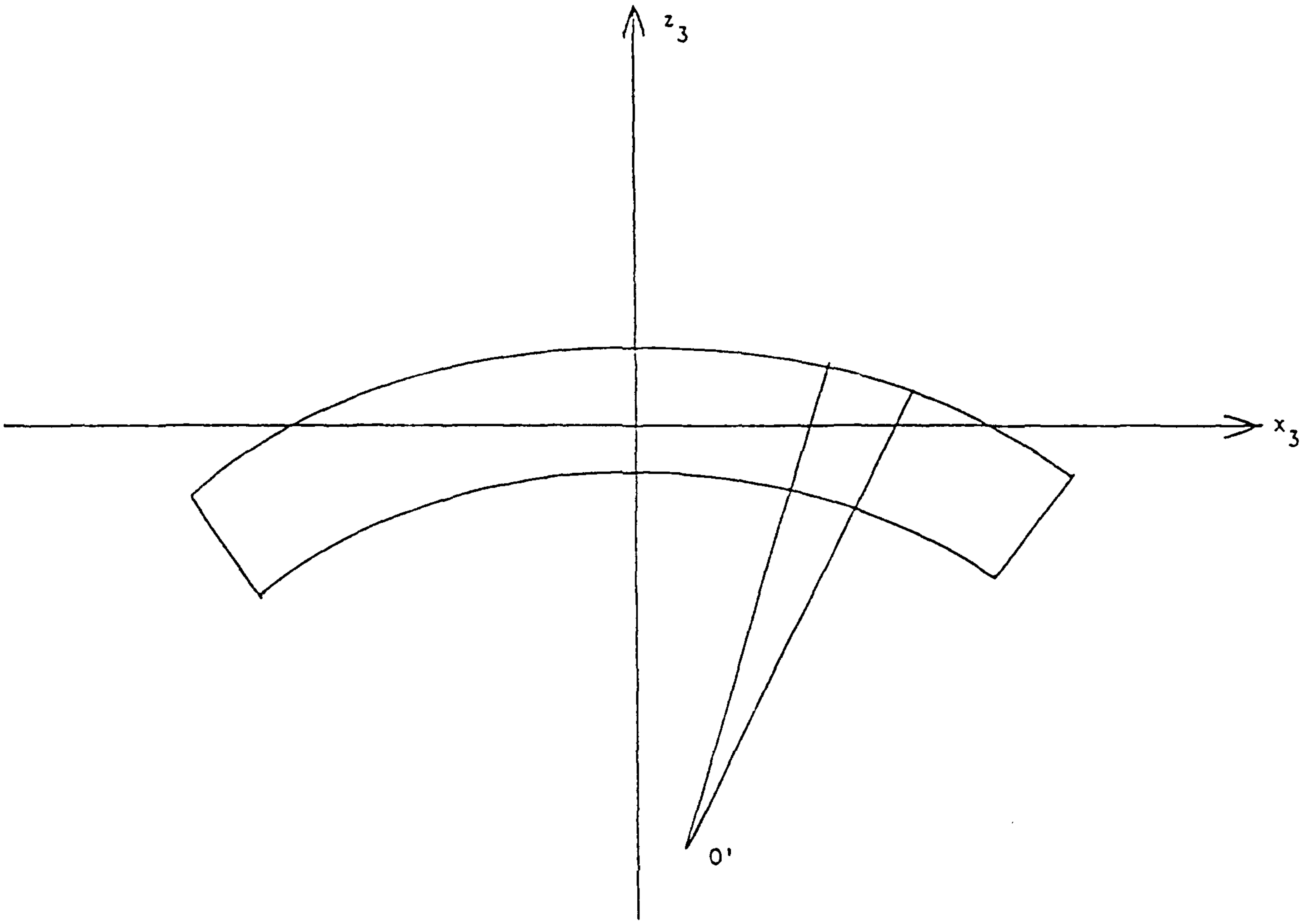


Figure 5.c
A section of a slice of the rod,
a cross-section perpendicular to the y_3 axis,
after bending.

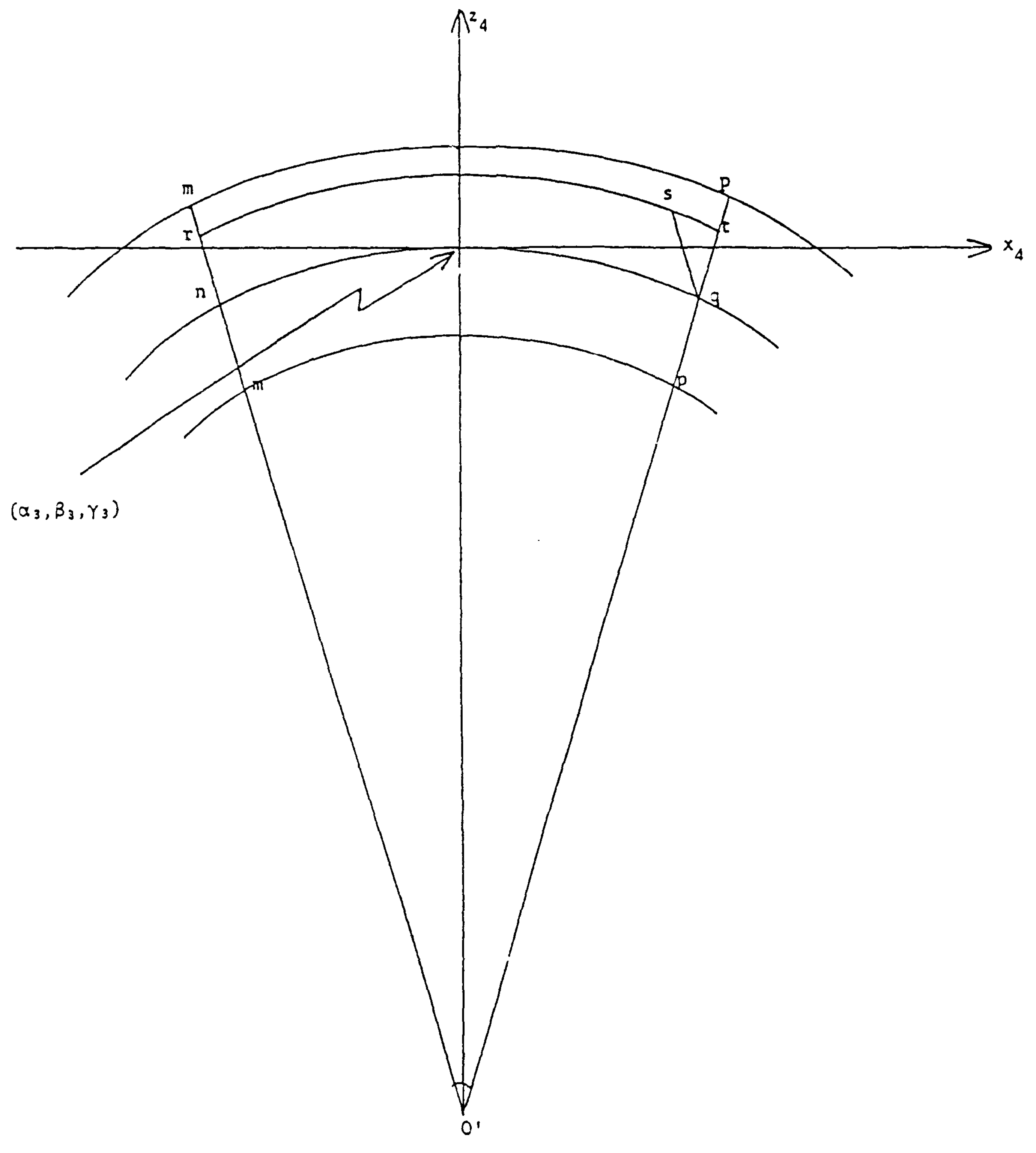


Figure 5.d
An enlergement of Figure 5.c

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